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ROUTH'S EQUATIONS AND VARIATIONAL PRINCIPLES†

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Holonomic mechanical systems with n degrees of freedom are considered in Routh variables, and the equations of motion consist of $k < n$ Lagrange-type equations and $2(n-k)$ Hamilton-type equations. Expressions are presented in Routh variables for the D'Alembert–Lagrange, Hamilton–Ostrogradskii and Hamilton (third form) variational principles, as well as the Hölder principle and the principle of least action in its Lagrange and Jacobi forms.

1. THE D'ALEMBERT–LAGRANGE PRINCIPLE AND ROUTH'S EQUATIONS

The fundamental variables characterizing the state of a holonomic mechanical system with n degrees of freedom at a certain time t are usually either Lagrange variables – generalized coordinates and velocities q_i and $\dot{q}_i = dq_i/dt$ $(i = 1, \ldots, n)$, or Hamilton variables – generalized coordinates and momenta q_i and $p_i = \partial L/\partial \dot{q}_i$ $(i = 1, \ldots, n)$, where $L(t, q, \dot{q}) = T(t, q, \dot{q}) + U(t, q)$ is the Lagrangian, T is the kinetic energy and U is the force function of the applied forces.

Routh [1] proposed taking as fundamental variables part of the Lagrange variables q_j, $\dot{q}_j$ $(j = 1, \ldots, k < n)$ and part of the Hamilton variables p_s $(s = k + 1, \ldots, n)$.

Let the kinetic energy of the system be a positive-definite quadratic form

$$T(q,\dot{q}) = \frac{1}{2}\sum_{i,j=1}^{n} a_{ij}(q_1,\ldots,q_n)\dot{q}_i\dot{q}_j \tag{1.1}$$

Then

$$p_s = \sum_{i=1}^{n} a_{si}\dot{q}_i, \quad s = k+1,\ldots,n \tag{1.2}$$

Since

$$D = \det(\partial^2 L/\partial \dot{q}_s \partial \dot{q}_r)_{s,r=k+1}^{n} \neq 0$$

all the velocities $\dot{q}_s$ may be expressed through Eqs (1.2) in terms of p_s and $\dot{q}_j$, and one obtains [2]

$$\dot{q}_s = \sum_{r=k+1}^{n} b_{sr}p_r - \sum_{j=1}^{k} \gamma_{sj}\dot{q}_j, \quad s = k+1,\ldots,n \tag{1.3}$$

where

$$b_{sr} = \frac{A_{rs}}{D}, \quad \gamma_{sj} = \sum_{r=k+1}^{n} b_{sr}a_{rj}; \quad j = 1,\ldots k; \quad s = k+1,\ldots,n$$

and A_{rs} is the cofactor of the element a_{rs} of the determinant D. Substituting expressions (1.3) into relation (1.1) we obtain an expression for the kinetic energy of system (1.1) in Routh variables

$$T^*(q_i,\dot{q}_j,p_s) = \frac{1}{2}\sum_{i,j=1}^{k} a_{ij}^*\dot{q}_i\dot{q}_j + \frac{1}{2}\sum_{r,s=k+1}^{n} b_{rs}p_rp_s \tag{1.4}$$

which does not contain terms with products of $\dot{q}_j$ and p_s, where

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$$a_{ij}^* = a_{ij} - \sum_{r,s=k+1}^{n} b_{rs} a_{rj} a_{si}; \quad i,j = 1,\dots,k$$

Both quadratic forms on the right-hand side of Eq. (1.4) are positive-definite.
Routh's function is defined by the formula

$$R(t, q_i, \dot{q}_j, p_s) = L(t, q_i, \dot{q}_i) - \sum_{s=k+1}^{n} \dot{q}_s p_s \tag{1.5}$$

in whose right-hand side all velocities $\dot{q}_s$ are expressed in terms of $\dot{q}_j$, p_s, so that

$$R(t, q_i, \dot{q}_j, p_s) = \frac{1}{2}\sum_{i,j=1}^{k} a_{ij}^* \dot{q}_i \dot{q}_j + \sum_{j=1}^{k} \sum_{s=k+1}^{n} \gamma_{sj} \dot{q}_j p_s - \frac{1}{2}\sum_{r,s=k+1}^{n} b_{rs} p_r p_s + U = \sum_{s=0}^{2} R_s \tag{1.6}$$

where

$$R_2 = \frac{1}{2}\sum_{i,j=1}^{k} a_{ij}^* \dot{q}_i \dot{q}_j, \quad R_1 = \sum_{j=1}^{k} \sum_{s=k+1}^{n} \gamma_{sj} \dot{q}_j p_s, \quad R_0 = U - \frac{1}{2}\sum_{r,s=k+1}^{n} b_{rs} p_r p_s$$

Equating the variations of both sides of Eq. (1.5), we obtain the relations

$$\frac{\partial R}{\partial q_i} = \frac{\partial L}{\partial q_i}, \quad \frac{\partial R}{\partial \dot{q}_j} = \frac{\partial L}{\partial \dot{q}_j}, \quad \frac{\partial R}{\partial p_s} = -\dot{q}_s; \quad i = 1,\dots,n; \quad j = 1,\dots,k; \quad s = k+1,\dots,n \tag{1.7}$$

using which we can express the general equation of dynamics in terms of Routh variables

$$\sum_{j=1}^{k} \left(\frac{d}{dt}\frac{\partial R}{\partial \dot{q}_j} - \frac{\partial R}{\partial q_j} - Q_j \right) \delta q_j + \sum_{s=k+1}^{n} \left(\frac{dp_s}{dt} - \frac{\partial R}{\partial q_s} - Q_s \right) \delta q_s = 0 \tag{1.8}$$

where $Q_i(t, q_i, \dot{q}_j, p_s)$ $(i = 1, \dots, n)$ are generalized non-potential forces. Equation (1.8) will hold whatever the virtual displacements δq_i, which are independent undefined quantities.

Relation (1.8) expresses the D'Alembert–Lagrange differential variational principle in terms of Routh variables.

Since the variations δq_i are arbitrary and independent, Eqs (1.8), taken together with the last group of Eqs (1.7), yield Routh's equations of motion

$$\frac{d}{dt}\frac{\partial R}{\partial \dot{q}_j} - \frac{\partial R}{\partial q_j} = Q_j, \quad j = 1,\dots,k \tag{1.9}$$

$$\frac{dp_s}{dt} = \frac{\partial R}{\partial q_s} + Q_s, \quad \frac{dq_s}{dt} = -\frac{\partial R}{\partial p_s}; \quad s = k+1,\dots,n \tag{1.10}$$

Routh's equations consist of k Lagrange-type second-order differential equations and $2(n - k)$ Hamilton type first-order differential equations. Of course, the system of Routh equations is equivalent to the systems of both Lagrange equations and Hamilton equations. Which of these systems is actually used is generally unessential though there may be various considerations in favour of the use of one system of equations or another.

When Routh's function $R(q_i, \dot{q}_j, p_s)$ does not depend explicitly on time and there are no non-potential forces, that is, $Q_i = 0 (i = 1, \dots, n)$, Eqs (1.9) and (1.10) have an energy integral

$$\sum_{i=1}^{k} \dot{q}_j \frac{\partial R}{\partial \dot{q}_j} - R = R_2 - R_0 = T^* - U = h = \text{const} \tag{1.11}$$

as is easily verified, either by multiplying each of Eqs (1.9) by $\dot{q}_j$, summing the results over all $j = 1, \dots, k$ and using Eqs (1.10), or by differentiating the function $R(q_i, \dot{q}_j, p_s)$ with respect to time along trajectories of Eq. (1.10) and using Eqs (1.9). Different expressions for integral (1.11) are obtained by using formulae (1.4) and (1.6).

If the function R does not depend explicitly on some of the coordinates q_r and the corresponding generalized forces Q_r vanish, while the expressions for the other generalized forces are independent of q_r, such coordinates are called cyclic – corresponding to them we have first integrals of Eqs (1.9) and (1.10)

$$p_r = c_r \tag{1.12}$$

where c_r are arbitrary constants.

Note that if the coordinates $q_s (s = k+1, \ldots, n)$ are cyclic, then the energy integral (1.11) of the system may be treated in the same way as the energy integral of the Lagrange subsystem (1.9) with kinetic energy $R_2(q_j, \dot{q}_j)$ and potential energy $-R_0(q_j, c_s)$, where the quadratic form

$$\frac{1}{2}\sum_{r,s=k+1}^{n} b_{rs}c_r c_s$$

occurring in the latter is the kinetic energy of the Hamilton subsystem (1.10). This treatment corresponds to Hertz's conception [3] of the kinetic origin of potential energy.

2. THE HAMILTON–OSTROGRADSKII PRINCIPLE. THE THIRD FORM OF HAMILTON'S PRINCIPLE

We will now derive the integral variational principles in Routh variables. We will integrate Eq. (1.8) with respect to t within certain arbitrarily chosen limits t_0 and t_1.

Integrating by parts the terms containing time derivatives and taking the second group of Eqs (1.10) into consideration, we obtain the relation

$$\left(\sum_{j=1}^{k}\frac{\partial R}{\partial \dot{q}_j}\delta q_j + \sum_{s=k+1}^{n} p_s\delta q_s\right)_{t_0}^{t_1} - \int_{t_0}^{t_1}\left[\sum_{j=1}^{k}\left(\frac{\partial R}{\partial q_j}\delta q_j + \frac{\partial R}{\partial \dot{q}_j}\delta \dot{q}_j\right) + \right.$$
$$\left. + \sum_{s=k+1}^{n}\left(\frac{\partial R}{\partial q_s}\delta q_s + \frac{\partial R}{\partial p_s}\delta p_s + p_s\delta\dot{q}_s + \dot{q}_s\delta p_s\right) + \sum_{i=1}^{n} Q_i\delta q_i\right]dt = 0$$

which leads to the equality

$$\int_{t_0}^{t_1}\left[\delta\left(R + \sum_{s=k+1}^{n}\dot{q}_s p_s\right) + \sum_{i=1}^{n} Q_i\delta q_i\right]dt = 0; \quad \delta q_i = 0 \quad \text{for} \quad t = t_0, t_1; \quad i = 1,\ldots,n \tag{2.1}$$

This equality is the expression in Routh variables of the Hamilton–Ostrogradskii variational principle, which is a necessary and sufficient condition for the motion of a system to be possible under the action of applied forces.

In the Hamilton–Ostrogradskii principle, the actual motion is compared with varied motions on the assumption that the system has the same configurations at the initial and final instants of time.

We will show that condition (2.1) yields Routh's equations. When doing so, however, some caution is necessary, since the variations of q_j and $\dot{q}_j$ cannot be regarded as independent [4]: if variations δq_j of class C_2 are given at each instant of time, then the variations $\delta\dot{q}_j$ are defined by the equations

$$\delta\dot{q}_j = \frac{d}{dt}\delta q_j, \quad j = 1,\ldots,k$$

Let us characterize the motion of the system by the displacement of a representative point in the $(2n+1)$-dimensional extended phase space of the variables $q_1, \ldots, q_n, \omega_1, \ldots, \omega_k, p_{k+1}, \ldots, p_n, t$. Condition (2.1) implies that

$$\int_{t_0}^{t_1}\left\{\delta\left[R(t, q_i, \omega_j, p_s) + \sum_{s=k+1}^{n}\dot{q}_s p_s\right] + \sum_{i=1}^{n} Q_i\delta q_i\right\}dt = 0;$$
$$\delta q_i = 0 \quad \text{for} \quad t = t_0, t_1; \quad i = 1,\ldots,n$$

in the family of curves $q_i(t)$, $\omega_j(t)$, $p_s(t)$ satisfying the differential equations

$$\dot{q}_j = \omega_j, \quad j = 1, \ldots, k \tag{2.2}$$

where the values of q_i and t at the initial and final points are fixed, while those of ω_j and p_s remain free.

By the rule of Lagrange multipliers

$$\int_{t_0}^{t_1} \left\{ \delta\left[R(t, q_i, \omega_j, p_s) + \sum_{s=k+1}^{n} \dot{q}_s p_s \right] + \sum_{i=1}^{n} Q_i \delta q_i + \sum_{j=1}^{k} \lambda_j (\dot{q}_j - \omega_j) \right\} dt = 0$$

for arbitrary variations of the variables q_i, ω_j, p_s, where the functions $\lambda_j(t)$ have to be determined. Varying and integrating by parts, we obtain

$$\int_{t_0}^{t_1} \left[\sum_{i=1}^{n} \frac{\partial R}{\partial q_i} \delta q_i + \sum_{j=1}^{k} \frac{\partial R}{\partial \omega_j} \delta \omega_j + \sum_{s=k+1}^{n} \left(\frac{\partial R}{\partial p_s} \delta p_s + \dot{q}_s \delta p_s + p_s \delta \dot{q}_s \right) + \sum_{i=1}^{n} Q_i \delta q_i + \right.$$

$$\left. + \sum_{j=1}^{k} \lambda_j (\delta \dot{q}_j - \delta \omega_j) \right] dt = \left(\sum_{j=1}^{k} \lambda_j \delta q_j + \sum_{s=k+1}^{n} p_s \delta q_s \right)_{t_0}^{t_1} + \int_{t_0}^{t_1} \left[\sum_{j=1}^{k} \left(\frac{\partial R}{\partial q_j} - \dot{\lambda}_j + Q_j \right) \delta q_j + \right.$$

$$\left. + \sum_{j=1}^{k} \left(\frac{\partial R}{\partial \omega_j} - \lambda_j \right) \delta \omega_j + \sum_{s=k+1}^{n} \left(\frac{\partial R}{\partial p_s} + \dot{q}_s \right) \delta p_s + \sum_{s=k+1}^{n} \left(\frac{\partial R}{\partial q_s} - \frac{dp_s}{dt} + Q_s \right) \delta q_s \right] dt = 0$$

The conditions for integral (2.1) to vanish may be written as follows, in view of (2.2)

$$\frac{d\lambda_j}{dt} = \frac{\partial R}{\partial q_j} + Q_j, \quad \lambda_j = \frac{\partial R}{\partial \dot{q}_j}, \quad \frac{dp_s}{dt} = \frac{\partial R}{\partial q_s} + Q_s, \quad \frac{dq_s}{dt} = -\frac{\partial R}{\partial q_s}$$

from which Routh's equations (1.9) and (1.10) follow,

When there are no non-potential forces, $Q_i = 0 (i = 1, \ldots, n)$, the third form of Hamilton's principle follows from the relation (2.1)

$$\delta \int_{t_0}^{t_1} \left(R + \sum_{s=k+1}^{n} \dot{q}_s p_s \right) dt = 0; \quad \delta q_i = 0 \quad \text{for} \quad t = t_0, t_1; \quad i = 1, \ldots, n \tag{2.3}$$

unlike the first form in Lagrange variables and the second form in Hamilton variables, respectively

$$\delta \int_{t_0}^{t_1} L dt = 0 \quad \text{and} \quad \delta \int_{t_0}^{t_1} \left(\sum_{i=1}^{n} \dot{q}_i p_i - H \right) dt = 0; \quad \delta q_i = 0 \quad \text{for} \quad t = t_0, t_1; \quad i = 1, \ldots, n \tag{2.4}$$

where the Hamiltonian is

$$H(t, q_i, p_i) = \sum_{i=1}^{n} \dot{q}_i p_i - L(t, q_i, \dot{q}_i) \tag{2.5}$$

For actual motion along the so-called "direct" path, along which the system may move in a given force field, the functions $q_i(t) (i = 1, \ldots, n)$ and $p_s(t) (s = k + 1, \ldots, n)$ satisfy Routh's equations (1.9) and (1.10). All other sufficiently close paths passing through two given points of the configuration space are known as "indirect" paths. It was proved above that for the direct path the Hamilton action in Routh variables

$$\int_{t_0}^{t_1} \left(R + \sum_{s=k+1}^{n} \dot{q}_s p_s \right) dt$$

has a stationary value compared with the indirect paths.

The converse is also true: if there is some path for which Eq. (2.3) holds, that path must be direct. Indeed, as shown previously for (2.1), one can similarly derive Eqs (1.9) and (1.10) for $Q_i = 0 (i = 1, \ldots, n)$ from condition (2.3).

We also note that Eqs (1.9) and (1.10) for $Q_i = 0 (i = 1, \ldots, n)$ are Euler's equations of variational problem (2.3) in Routh variables.

At first sight, one can show, using Eqs (2.5) and (1.5), that the third form (2.3) of Hamilton's principle in no way differs from the first and second form (2.4) of the principle. However, this is true only for paths $q_i = q_i(t)$ and $p_i = p_i(t)$ such that the functions $q_i(t)$ and $p_i(t)$ satisfy the equations $p_i = \partial L/\partial \dot{q}_i$ $(i = 1, \ldots, n)$, but in the general case this is not so [2].

The following difference between variational problems (2.3) and (2.4) must also be borne in mind. In the first form of the principle, the curves $q_i(t) (i = 1, \ldots, n)$ are considered in the extended $(n + 1)$-dimensional coordinate space passing through two given points $A_0(t_0, q_i^0)$ and $A_1(t_1, q_i^1)$, where the initial and final times t_0 and t_1, as well as the initial and final positions of the system, q_i^0 and q_i^1, are fixed in advance. In the second and third forms of the principle, admissible indirect paths are sufficiently close arbitrary curves in the $(2n + 1)$-dimensional extended phase space of the variables t, q_i, p_i (in the second form) or the variables t, q_i, $\dot{q}_j$, p_s (in the third form) passing through the points B_0 and B_1 or C_0 and C_1, respectively, at times t_0 and t_1, for fixed initial and final values of the variables t and q_i and arbitrary values of $\dot{q}_j$, p_s. These curves may not, in general, satisfy the relations $p_i = \partial L/\partial \dot{q}_i$. These three integrals have the same values for the actual motion in each specific case, but while the integral of the first form may reach a minimum, the integrals of the second and third forms for the same problem may have neither maximum nor minimum [4]. It should also be noted that, unlike the points A_0 and A_1, the points B_0 and B_1 or the points C_0 and C_1 cannot be chosen arbitrarily, since in the general case a direct path cannot be defined through two arbitrary points of the extended phase space: the points B_0 and B_1 or C_0 and C_1 are chosen on the same direct path for which Hamilton's principle is formulated [2].

Thus, principle (2.3), occupying a position intermediate between the forms (2.4), is of independent value because of the assumptions concerning the arbitrariness and independence of the variations δq_i and δp_s within the interval $[t_0, t_1]$.

When the coordinates q_s $(s = k + 1, \ldots, n)$ are cyclic, first integrals of the form (1.12) $p_s = c_s$ exist. If one then considers only variations that leave the generalized momenta $p_s = c_s$ constant, then, for fixed initial and final positions of the system,

$$\delta \int_{t_0}^{t_1} \sum_{s=k+1}^{n} \dot{q}_s p_s dt = \sum_{s=k+1}^{n} p_s \delta \int_{t_0}^{t_1} \dot{q}_s dt = 0$$

and Hamilton's principle (2.3) takes the following form [5, 1].

$$\delta \int_{t_0}^{t_1} R(t, q_j, \dot{q}_j, c_s) dt = 0; \quad \delta q_j = 0 \quad \text{for} \quad t = t_0, t_1; \quad j = 1, \ldots, k \tag{2.6}$$

3. HÖLDER'S PRINCIPLE

In principles (2.1), (2.3) and (2.4), we considered synchronous variations: a point P on the direct path at time t was associated with the point P' on the indirect path corresponding to the same instant of time. This was possible because motions along the direct and indirect paths took place within the same time interval $t_1 - t_0$.

We will now consider asynchronous variation, when a point q_i $(i = 1, \ldots, n)$ on the actual trajectory at time t is associated with a point $q_i + \delta q_i$ on the varied trajectory at time $t + \delta t$. The variations δq_i and δt are assumed to be functions of time of class C_2, and moreover the relations between the Cartesian and generalized coordinates of the system to not involve the time t.

Let us evaluate the integral on the left-hand side of (2.1). Using the formula

$$\delta \frac{dq_i}{dt} = \frac{d}{dt}\delta q_i - \dot{q}_i \frac{d}{dt}\delta t, \quad i = 1, \ldots, n$$

and integrating by parts, we get

$$\int_{t_0}^{t_1} \left[\delta\left(R + \sum_{s=k+1}^{n} \dot{q}_s p_s \right) + \sum_{i=1}^{n} Q_i \delta q_i \right] dt = \left(\sum_{j=1}^{k} \frac{\partial R}{\partial \dot{q}_j} \delta q_j + \sum_{s=k+1}^{n} p_s \delta q_s \right)_{t_0}^{t_1} +$$

$$
+\int_{t_0}^{t_1}\left[\frac{\partial R}{\partial t}\delta t-\sum_{j=1}^{k}\left(\frac{d}{dt}\frac{\partial R}{\partial \dot q_j}-\frac{\partial R}{\partial q_j}-Q_j\right)\delta q_j-\sum_{s=k+1}^{n}\left(\frac{dp_s}{dt}-\frac{\partial R}{\partial q_s}-Q_s\right)\delta q_s+\right.
$$

$$
\left.+\sum_{s=k+1}^{n}\left(\frac{dq_s}{dt}+\frac{\partial R}{\partial p_s}\right)\delta p_s-\left(\sum_{j=1}^{k}\dot q_j\frac{\partial R}{\partial \dot q_j}+\sum_{s=k+1}^{n}\dot q_s p_s\right)\frac{d}{dt}\delta t\right]dt \tag{3.1}
$$

If the variations δq_i $(i = 1, \dots, n)$ are virtual displacements at each time t, where $\delta q_i = 0$ at times t_0 and t_1, then, by Eqs (1.9) and (1.10), we obtain the equality

$$
\int_{t_0}^{t_1}\left[\delta\left(R+\sum_{s=k+1}^{n}\dot q_s p_s\right)+\left(\sum_{j=1}^{k}\dot q_j\frac{\partial R}{\partial \dot q_j}+\sum_{s=k+1}^{n}\dot q_s p_s\right)\frac{d}{dt}\delta t-\frac{\partial R}{\partial t}\delta t+\sum_{i=1}^{n}Q_i\delta q_i\right]dt=0
$$

$$
\delta q_i = 0 \quad \text{for} \quad t=t_0, t_1; \quad i=1,\dots,n \tag{3.2}
$$

which expresses Hölder's principle [4, 6] in Routh variables. It holds provided that the virtual displacements $\delta q_i \in C_2$ relative to the actual motion exist at each instant of time and vanish at times t_0 and t_1, while the functions $\delta t \in C_2$ do not necessarily vanish at times t_0 and t_1.

Note that in the case of synchronous variations, when $\delta t \equiv 0$, equality (3.2) implies the Hamilton–Ostrogradskii principle (2.1) and, additionally, if also $Q_i = 0 (i = 1, \dots, n)$, it implies Hamilton's principle (2.3).

4. LAGRANGE'S PRINCIPLE

Let us assume that Routh's function $R(q_i, \dot q_j, p_s)$ does not depend explicitly on time, and that there are no non-potential forces Q_i, so that energy integral (1.11) exists. The system itself can chose its motions from motions with a given reserve h of total energy, so that we need only compare trajectories that satisfy condition (1.11) [7].

Given the conditions $\partial R/\partial t \equiv 0$, $Q_i = 0 (i = 1, \dots, n)$, Hölder's principle (3.2) takes the form

$$
\int_{t_0}^{t_1}\left[\delta\left(R+\sum_{s=k+1}^{n}\dot q_s p_s\right)+\left(\sum_{j=1}^{k}\dot q_j\frac{\partial R}{\partial \dot q_j}+\sum_{s=k+1}^{n}\dot q_s p_s\right)\frac{d}{dt}\delta t\right]dt=0 \tag{4.1}
$$

$$
\delta q_i = 0 \quad \text{for} \quad t=t_0, t_1; \quad i=1,\dots,n
$$

Integral (1.11) implies the relation

$$
R=\sum_{j=1}^{k}\dot q_j\frac{\partial R}{\partial \dot q_j}-h
$$

Taking this into consideration, we can write (4.1) in the form

$$
\int_{t_0}^{t_1}\left[\delta\left(\sum_{j=1}^{k}\dot q_j\frac{\partial R}{\partial \dot q_j}+\sum_{s=k+1}^{n}\dot q_s p_s\right)+\left(\sum_{j=1}^{k}\dot q_j\frac{\partial R}{\partial \dot q_j}+\sum_{s=k+1}^{n}\dot q_s p_s\right)\frac{d}{dt}\delta t\right]-(t_1-t_0)\delta h=0
$$

whence we obtain Lagrange's principle of least action in Routh variables for $\delta h = 0$ and fixed end points (in q-space)

$$
\delta\int_{t_0}^{t_1}\left(\sum_{j=1}^{k}\dot q_j\frac{\partial R}{\partial \dot q_j}+\sum_{s=k+1}^{n}\dot q_s p_s\right)dt=0;\quad \delta q_i=0 \quad \text{for} \quad t=t_0,t_1;\quad i=1,\dots,n;\quad \delta h=0 \tag{4.2}
$$

The actual motion of a conservative holonomic system between two given configurations differs from the kinematically possible motions that can take place between the same configurations and with the same total energy h, in that for actual motion the total variation of the Lagrange action

$$\int_{t_0}^{t_1}\left(\sum_{j=1}^{k}\dot q_j\frac{\partial R}{\partial \dot q_j}+\sum_{s=k+1}^{n}\dot q_s p_s\right)dt \tag{4.3}$$

has a stationary value.

Note that, as a consequence of the energy integral, the time in which the system transfers from one position to another depends on the path and is determined by it, so that the upper limit t_1 in integrals (4.2) and (4.3) is variable and the variations of integrals (4.2) and (4.3) must be total.

Taking relations (1.4), (1.6) and (1.7) into account, we have the following obvious equality

$$\sum_{j=1}^{k}\dot q_j\frac{\partial R}{\partial \dot q_j}+\sum_{s=k+1}^{n}\dot q_s p_s=2T^* \tag{4.4}$$

as a consequence of which Lagrange's principle (4.2) may also be expressed in the form

$$\delta\int_{t_0}^{t_1} 2T^* dt=0;\quad \delta q_i=0\quad \text{for}\quad t=t_0,t_1;\quad i=1,\dots,n;\quad \delta h=0 \tag{4.5}$$

which is similar to its form in Lagrange variables ($T = T^*$).

The principle of least action, like Hamilton's principle, expresses a necessary and sufficient condition of actual motion, and it may be used to derive the equations of motion. Indeed, let us construct Lagrange's function with multiplier λ for the conditional variational problem (4.5) [7]

$$F=2T^*+\lambda(T^*-U-h)$$

The transversality condition for a sliding end-point at the upper limit of the integral

$$F-\sum_{i=1}^{n}\dot q_i\frac{\partial F}{\partial \dot q_i}=0$$

leads to the equality $(1 + \lambda)T^* = 0$, from which we find $\lambda = -1$ and, bearing (1.5) in mind, we obtain the function

$$F=R+\sum_{s=k+1}^{n}\dot q_s p_s+h$$

Consequently, Euler's equations for the variational problem with integrand F, considered in Routh variables, have the same form as Eqs (1.9) and (1.10) with $Q_i = 0$ $(i = 1, \dots, n)$.

When the coordinates q_s $(s = k + 1, \dots, n)$ are cyclic and first integrals of the form (1.12) $p_s = c_s$ exist, Lagrange's principle of least action, considered for variations that leave the values of the momenta p_s constant, and on the assumption that the initial and final positions of the system are fixed, takes the form

$$\delta\int_{t_0}^{t_1}\sum_{j=1}^{k}\dot q_j\frac{\partial R}{\partial \dot q_j}dt=0,\quad R=R(q_j,\dot q_j,c_s) \tag{4.6}$$
$$\delta h=0,\quad \delta q_j=0\quad \text{for}\quad t=t_0,t_1;\quad j=1,\dots,k$$

4. JACOBI'S PRINCIPLE

Using the energy integral, Jacobi [8] eliminated the time from Lagrange's principle and reduced everything to spatial elements, thus placing the principle of least action in a geometrical context.

The energy integral and the expression for the Lagrange action are

$$T=U+h=\sqrt{T(U+h)},\quad \int_{t_0}^{t_1}2\sqrt{T(U+h)}dt \tag{5.1}$$

Owing to the existence of the energy integral, there is a complicated relation between the variations of the variables t and q_i, $\dot{q}_i$ [9]. To avoid this difficulty, we choose a new independent variable τ whose values vary between constant limits τ_0 and τ_1, independently of time. For example, one can take as this variable one of the coordinates q_i, which is a monotone function of t in the interval under consideration [8, 4]. During the motion of the system, $q_i (i = 1, \ldots, n)$ will be functions of the variable τ, whose derivatives with respect to that variable will be denoted by $q_i' = dq_i/d\tau$.

Taking equality (1.1) into consideration, we put [9]

$$2\tilde{T}(q_i, q_i') = \sum_{i,j=1}^{n} a_{ij} q_i' q_j'$$

We then have

$$T = \tilde{T}\left(\frac{d\tau}{dt}\right)^2, \quad \tilde{T}\left(\frac{d\tau}{dt}\right)^2 = U + h; \quad \frac{d\tau}{dt} = \sqrt{\frac{U+h}{\tilde{T}}} \tag{5.2}$$

Substituting the last expression of (5.2) into the action (5.1) and using Eq. (4.5), we obtain Jacobi's principle of least of action in Lagrange variables

$$\delta \int_{\tau_2}^{\tau_1} 2\sqrt{\tilde{T}(U+h)}d\tau = 0; \quad \delta q_i = 0 \text{ for } \tau = \tau_0, \tau_1; \quad i = 1,, \ldots, n; \quad \delta h = 0 \tag{5.3}$$

which is geometric in nature [7, 9].

In actual motion, the Jacobi action takes a stationary value compared with its values for infinitely close neighbouring motions that take the system from the same initial position to the same final position, on the assumption that Eqs (5.1) remain valid with the same value of the constant h as in the actual motion.

The problem of determining the trajectory of the representative point in q-space is thus reduced to problem (5.3) of variational calculus with fixed end-points. The velocity of motion of the representative point along the trajectories is found from the energy integral [7, 9].

Finally, we will express Jacobi's principle in Routh variables. Comparing principle (5.3) with the first form (2.4) of Hamilton's principle, we conclude that the integrand in (5.3) may be taken as a new Lagrangian $\tilde{L}(q_i, q_i')$ with independent variable τ instead of the time t and velocities q_i' [4]. By analogy with the function (1.5), we introduce a new Routh function

$$\tilde{R}(q_i, q_j', \tilde{p}_s) = \tilde{L}(q_i, q_i') - \sum_{s=k+1}^{n} q_s' \tilde{p}_s, \quad \tilde{L}(q_i, q_i') = 2\sqrt{\tilde{T}(U+h)} \tag{5.4}$$

where the momenta are

$$\tilde{p}_s = \frac{\partial \tilde{L}}{\partial q_s'} = \frac{\partial \tilde{T}}{\partial q_s'}\sqrt{\frac{U+h}{\tilde{T}}}, \quad s = k+1, \ldots, n$$

Comparing the variations of both sides of Eq. (5.4) we find relations analogous to Eqs (1.7).

As a result, we obtain an expression for Jacobi's principle in Routh variables:

$$\delta \int_{\tau_0}^{\tau_1} \left(\tilde{R}(q_i, q_j', \tilde{p}_s) + \sum_{s=k+1}^{n} q_s' \tilde{p}_s \right) d\tau = 0; \quad \delta q_i = 0 \quad \text{for} \quad \tau = \tau_0, \tau_1; \quad \delta h = 0 \tag{5.5}$$

The equations of the extremals of problem (5.5) are Routh's equations

$$\frac{d}{d\tau}\frac{\partial \tilde{R}}{\partial q_j'} = \frac{\partial \tilde{R}}{\partial q_j}, \quad \frac{d\tilde{p}_s}{d\tau} = \frac{\partial \tilde{R}}{\partial q_s}, \quad q_s' = -\frac{\partial \tilde{R}}{\partial \tilde{p}_s}; \quad j = 1, \ldots, k; \quad s = k+1, \ldots, n \tag{5.6}$$

from which, returning to the independent variable t via the last relations in (5.2), we obtain Lagrange's equations and the equivalent equations (1.9) and (1.10) with $Q_i = 0 (i = 1, \ldots, n)$.

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